

Energy conservation for a radiating charge in classical electrodynamics

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Abstract

It is shown that the well-known disparity in classical electrodynamics between the power radiated in electromagnetic fields and the power-loss, as calculated from the radiation reaction on a charge undergoing a non-uniform motion, is successfully resolved when a proper distinction is made between quantities expressed in terms of a “real time” and those expressed in terms of a retarded time. It is shown that the expression for the real-time radiative power loss from a charged particle is somewhat different from the familiar Larmor’s formula, or in a relativistic case, from Liénard’s formula.

1 Introduction

One of the most curious and perhaps an equally annoying problem in classical electrodynamics is that the power emitted from an accelerated charge does not appear to exactly match with the radiation reaction on the charge. In the standard, Larmor's radiation formula (generalized to Liénard's formula in the case of a relativistic motion), the radiated power is directly proportional to the *square of acceleration* of the charged particle. From the energy conservation law, it is to be surely expected that the power emitted in radiation fields equals the power loss of the radiating charge. But from the radiation reaction equation, the power loss of a radiating charge is directly proportional to its *rate of change of acceleration* (see e.g., [1, 2, 3]). Although the two formulae do yield the same value of energy when integrated over a time interval chosen such that the scalar product of velocity and acceleration vectors of the charge is the same at the beginning as at the end of the interval (a periodic motion !), the two calculations do not match when the charge is still undergoing a non-uniform, non-periodic motion at either end of the time interval [1]. In any case the functional forms of the two formulae appear totally different. This puzzle has defied a satisfactory solution despite the continuous efforts for the last 80 years or so. It is generally thought that the root-cause of this problem may lie in the radiation-reaction equation, whose derivation is considered to be not as rigorous as that of the formula for power radiated. Some interesting proposals for the “removal” of the above discrepancy include the ad hoc assumption of an acceleration dependent term either in a modified form of the Lorentz-force formula [4], or in the radiation-reaction equation in the form of a “bound-energy” term for an accelerated charge [5, 6] (also called as an internal-energy term or simply a “Schott-term”, based on the first such suggestion by Schott [7]), or a somewhat related proposition [8] that even the *proper-mass* of a radiating charged particle (e.g., of an

electron) varies, or even a combination of some of these propositions [9, 10]. Alternately it has been suggested [11] that there may be some fundamental difference in the electrodynamics of a continuous charge distribution and of a “point charge” (with a somewhat different stress-energy tensor for the latter). It is interesting to note that an understanding of this anomaly has sometimes been sought beyond the border of the classical electrodynamics (e.g., in the vacuum-fluctuations of the electromagnetic fields in quantum theory [12]). Ideally one would expect the classical electrodynamics to be mathematically consistent *within itself*, even though it might not be adequate to explain all experimental phenomena observed for an elementary charged particle. Because of the vastness of literature on this subject, we refer the reader to a recent review article [14] for further references on these and other interesting ideas that have been proposed to remove this seemingly inconsistency in classical electrodynamics.

We intend to show here that the difference perceived in the two power formulae is merely a reflection of the fact that the two are calculated in terms of two different time systems. While the radiation reaction formula is expressed in terms of quantities being evaluated at the “real time” of the charged particle, Larmor’s radiation formula is written actually in terms of quantities evaluated at a retarded time. The difference between the two time systems is only $\sim r_0/c$ for a charged particle of radius r_0 , and is as such vanishingly small for a charge distribution that reduces to a “point” in the limit. But as we will show below, it still gives rise to a finite apparent difference, independent of r_0 , for the power calculations in the two formulae, because of the presence of the $1/r_0$ term in the self-field energy of the charge. By bringing out this simple relation between these two seemingly contradictory results, we demonstrate in this way their mutual consistency, without invoking any additional hypothesis.

2 The calculation of self-force

Poynting's theorem allows us to relate the rate of electromagnetic energy outflow through the surface boundary of a charge distribution to the rate of change in the mechanical energy of the enclosed charges. Accordingly, in order to compare the outflow of radiation (as implied by say, Larmor's formula) from a charged particle that may be undergoing a non-uniform motion, we need to evaluate the self-force "felt" by the charge. We consider a classical, spherical-shell model for the charge particle. We take the motion of the charged particle to be such that there is always some inertial frame available in which the whole charge is momentarily at rest, i.e., there is no differential motion between the various constituents of the charged particle in its instantaneous rest-frame (a "rigid-motion" though not a "rigid-body" motion [15]). We have to consider the force on one part of the charged sphere due to the fields from its all other parts, positions of the latter calculated at the retarded times, and then the force to be integrated over the whole sphere. By doing the force calculations in the instantaneous rest-frame of the charge, we can avoid the computations of magnetic fields. The electric field at the location of an element of charge de due to another element de' is given by [1],

$$d\mathbf{E} = de' \left[\frac{\mathbf{n} - \boldsymbol{\beta}}{r^2 \gamma^2 (1 - \boldsymbol{\beta} \cdot \mathbf{n})^3} + \frac{\mathbf{n} \times \{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{rc(1 - \boldsymbol{\beta} \cdot \mathbf{n})^3} \right]_{ret}, \quad (1)$$

where the quantities on the right hand side are to be evaluated at the retarded time. More specifically, $\boldsymbol{\beta} = \mathbf{v}/c$, $\dot{\boldsymbol{\beta}} = \dot{\mathbf{v}}/c$, and $\gamma = 1/\sqrt{1 - \beta^2}$ represent respectively the velocity, acceleration and the Lorentz factor of de' at the retarded time, and $\mathbf{r} = r\mathbf{n}$ is the radial vector from the retarded position of de' to the "present" position of de . While calculating the field arising from the whole sphere, one has to then also take into account that the retarded times are different for various charge elements (de') that belong to different parts of the sphere. It should be noted that in the case of an accel-

erated motion, even if there may be no motion between various parts of an extended object ($r_0 > 0$) in its instantaneous rest frame, there will be, in general, a differential acceleration between various parts of the object, as seen in other inertial frames. Such a differential acceleration, which occurs even in the case of a uniformly accelerated motion (see e.g., ref. [16]), actually represents the changing Lorentz contraction of an accelerated object. A proper account of the work done against the Coulomb forces of self-repulsion between various parts of a charged body during its changing Lorentz contraction, is necessary to get the correct formulae for the total energy in the Coulomb-fields of such a charge [17]. But such a differential motion between various parts of the charge, as seen in a relatively moving frame, can be ignored as far as its radiation field energy is concerned (see also the discussion in ref. [7]).

Actually the mathematical details of the calculations of self-force, carried out first by Lorentz [18] and done later more meticulously by Schott [7], are available in different forms in modern text-books [1, 2, 3, 13]. Such calculations are done mostly with respect to the instantaneous rest-frame of the charge, but the results derived can be written in a relativistic covariant form and then applied in any inertial frame. We will follow the approach of [2] and [3], where one starts directly from the electric fields (i.e., Eq. (1) here) instead of the potentials as in [1]. It is generally assumed in such calculations that the motion of the charged particle varies slowly so that during the light-travel time across the particle, any change in its velocity, acceleration and other higher time derivatives is relatively small. This is equivalent to the conditions that $|\dot{\mathbf{v}}|\tau \ll c$, $|\ddot{\mathbf{v}}|\tau \ll |\dot{\mathbf{v}}|$, etc., where $\tau = r_0/c$.

Under these conditions, one can make a Taylor series expansion of the retarded quantities in Eq. (1) around the present time t_0 ; the retarded time being related to the present time by $t' = t_0 - r/c$. Strictly speaking, r as well as \mathbf{r} in Eq. (1) do not represent the distance between de and de' at the present time and need to be measured from the

retarded position of de' [3]. But it turns out that any net effect of this subtle difference for a spherically symmetric system shows up only in terms of order higher than we are interested in and which become negligible for a small enough r_0 . Remembering that $\mathbf{v}(t_0) = 0$ (in the instantaneous rest frame), we can thus write,

$$\begin{aligned}\mathbf{v}(t') &= -\frac{\dot{\mathbf{v}}r}{c} + \frac{\ddot{\mathbf{v}}r^2}{2c^2} + \dots, \\ \dot{\mathbf{v}}(t') &= \dot{\mathbf{v}} - \frac{\ddot{\mathbf{v}}r}{c} + \dots, \\ \left[1 - \frac{\mathbf{v} \cdot \mathbf{n}}{c}\right]_{ret}^{-3} &= 1 - \frac{3\dot{\mathbf{v}} \cdot \mathbf{r}}{c^2} + \frac{3r(\ddot{\mathbf{v}} \cdot \mathbf{r})}{2c^3} + \dots,\end{aligned}$$

all quantities on the right hand side being evaluated at the present time. Substituting these expressions in Eq. (1), the electric force on de due to de' can be written as,

$$d\mathbf{f} = dede' \left\{ \left[\frac{\mathbf{r}}{r^3} + \frac{\dot{\mathbf{v}}}{rc^2} - \frac{\ddot{\mathbf{v}}}{2c^3} - \frac{3\mathbf{r}(\dot{\mathbf{v}} \cdot \mathbf{r})}{r^3c^2} + \frac{3\mathbf{r}(\ddot{\mathbf{v}} \cdot \mathbf{r})}{2r^2c^3} \right] + \left[\frac{\mathbf{r}(\dot{\mathbf{v}} \cdot \mathbf{r})}{r^3c^2} - \frac{\dot{\mathbf{v}}}{rc^2} - \frac{\mathbf{r}(\ddot{\mathbf{v}} \cdot \mathbf{r})}{r^2c^3} + \frac{\ddot{\mathbf{v}}}{c^3} \right] \right\}. \quad (2)$$

All other higher order terms will give negligible contribution for a small r/c . The total self-force on the spherical charge e can be found by integrating the above expression over both de and de' . It should be noted here that the net force contribution from two charge elements, when taken in pairs, does not cancel, i.e. the force on de' due to de is not equal and opposite to that on de due to de' . But one can exploit the spherical symmetry of the charge distribution to cancel some terms upon integration. This is due to the fact that the average of $\mathbf{r}(\dot{\mathbf{v}} \cdot \mathbf{r})$ for all possible angles over a spherically symmetric region is $r^2\dot{\mathbf{v}}/3$, while \mathbf{r} averages to zero [2].

Now we would like to emphasize two interesting points that do not seem to have been highlighted in such calculations before.

(i) The velocity-fields (terms contained within the first square bracket on the right hand side in Eq. (2)) contribute nothing to the electromagnetic self-force of a charge. The contributions of the various terms get cancelled when integrated over the whole

spherical distribution. Since the self-force from the velocity-fields is zero in the instantaneous rest frame of the charge, from the Lorentz transformation it is also zero in other inertial frames of reference. Of course a nil self-force from velocity fields is an obvious expectation in the case of a charge moving with a uniform velocity, where this is the only term in the electromagnetic field of the charge. But the contribution of velocity fields to the net self-force turns out to be nil even in the case of an accelerated charge, when the retarded time values are used in the field calculations for a spherical charge distribution.

(ii) The only non-zero terms that remain in the net self-force are,

$$\mathbf{f} = e^2 \left[-\frac{2\dot{\mathbf{v}}}{3r_0c^2} + \frac{2\ddot{\mathbf{v}}}{3c^3} \right], \quad (3)$$

which can be written as,

$$\mathbf{f} = -\frac{2e^2}{3r_0c^2}[\dot{\mathbf{v}} - \ddot{\mathbf{v}}\tau],$$

or

$$\mathbf{f} = -\frac{2e^2}{3r_0c^2}\dot{\mathbf{v}}_{ret}, \quad (4)$$

where $\dot{\mathbf{v}}_{ret} = \dot{\mathbf{v}} - \ddot{\mathbf{v}}\tau$ represents the acceleration that the charge had at a time $t_0 - \tau$ (to a first order in τ). Thus the “present” value of the net self-force (including the radiation-reaction drag force) “felt” by a spherical-shell charge, at any instant, turns out to be directly proportional to the acceleration that the charge (as a whole) was undergoing at a time τ earlier.

3 The rate of work done against self-force

To calculate the instantaneous rate of work being done against the self-force of a moving charge in an inertial frame, one has to take the scalar product of the *present value* of the self-force (which incidentally is proportional to a *retarded value* of the acceleration, Eq. (4)) and the *present* velocity, \mathbf{v} , of the charge, both measured in that frame. One

can derive the relevant formulae with respect to a frame in which the charge has a non-relativistic motion, and then use the condition of relativistic covariance to get the more general formulae valid for any inertial frame. For a non-relativistic motion, the expression for force can be used directly from that in the instantaneous rest frame (Eq. (4) above).

Accordingly, the rate of work done against self-fields of an accelerated charge is given by,

$$\frac{d\mathcal{E}}{dt} = \frac{2e^2}{3r_0c^2}(\dot{\mathbf{v}}_{ret} \cdot \mathbf{v}) , \quad (5)$$

which, equivalently, in terms of the “present” time (real-time) quantities is written as,

$$\frac{d\mathcal{E}}{dt} = \frac{2e^2}{3r_0c^2}(\dot{\mathbf{v}} \cdot \mathbf{v} - \ddot{\mathbf{v}} \cdot \mathbf{v} \frac{r_0}{c}) . \quad (6)$$

The more familiar form for this expression is [1, 2, 3],

$$\frac{d\mathcal{E}}{dt} = \frac{4U_0}{3c^2}\dot{\mathbf{v}} \cdot \mathbf{v} - \frac{2e^2}{3c^3}\ddot{\mathbf{v}} \cdot \mathbf{v} , \quad (7)$$

where $U_0 = e^2/2r_0$ represents the electromagnetic self-energy in Coulomb fields of a spherical-shell charge that is permanently stationary in an inertial frame. The first term on the right hand side in Eq. (7) represents the change in the self-Coulomb field energy of the charge as its velocity changes. This term when combined with the additional work done during a changing Lorentz contraction (not included in Eq. (7)) against the Coulomb self-repulsion force of the charged particle, on integration leads to the correct expression for energy in fields of a uniformly moving charged particle [17]. Thus it is only the second term on the right hand side of Eq. (7) that represents the “excess” power going into the electromagnetic fields of a charge with a non-uniform motion. It has always seemed enigmatic that if Larmor’s formula indeed represents the instantaneous rate of power loss for an accelerated charge, why the above term contains $-\ddot{\mathbf{v}} \cdot \mathbf{v}$ instead of $\dot{\mathbf{v}}^2$.

Now a point that does not seem to have been realized before is that if in Eq. (5) we kept the acceleration in terms of its value at the retarded time and instead express the velocity also in terms of its value at the retarded time ($t_0 - \tau$), then for $\mathbf{v} = \mathbf{v}_{ret} + \dot{\mathbf{v}}_{ret}\tau$ (to the required order in τ) we get,

$$\frac{d\mathcal{E}}{dt} = \frac{2e^2}{3r_0c^2}(\dot{\mathbf{v}}_{ret} \cdot \mathbf{v}_{ret} + \dot{\mathbf{v}}_{ret}^2 \tau) , \quad (8)$$

or

$$\frac{d\mathcal{E}}{dt} = \left[\frac{2e^2}{3r_0c^2} \dot{\mathbf{v}} \cdot \mathbf{v} + \frac{2e^2}{3c^3} \dot{\mathbf{v}}^2 \right]_{ret} . \quad (9)$$

Thus if we examine the rate at which energy is pouring into the electromagnetic fields of a charged particle at some given instant, then Eq. (7) gives the rate in terms of the real-time values of quantities specifying the motion of the charge at that particular instant. On the other hand, Eq. (9) yields the familiar Larmor's radiation formula (second term on the right hand side), but at a cost that a real-time rate of field energy outflow from the charge distribution is expressed in terms of quantities specifying the motion of the charge at a retarded time, and not in terms of their values at the present time. The time difference τ may be exceedingly small (infinitesimal in the limit $r_0 \rightarrow 0$), but still its effect in the energy calculations is finite because of the presence of the $1/r_0$ term for energy in self-fields outside the sphere of radius r_0 . Now τ is actually the time taken for a signal to reach from the centre of the sphere to the points at its surface. Essentially it implies that if the electromagnetic energy outflow through the boundary of the spherical charge distribution (as inferred from the rate of work done against the self-force of the charge distribution) is expressed in terms of the quantities describing the *retarded motion* of an equivalent "point" charge at the center of the sphere, we obtain the familiar Larmor's formula. We may also point out that even in the standard text-book statement of Larmor's formula for radiation from a non-relativistically moving point charge (see e.g. ref. [1]), the rate of energy flow

at a time t through a spherical surface of radius r_0 is written always in terms of the retarded value of the acceleration (at time $t - r_0/c$) of the point charge, the expression being exactly equal to the second term on the right hand side of Eq. (9).

This not only resolves the apparent “discrepancy” in the two power formulae, but also shows an intimate relation between the energy in the radiation fields and that in the Coulomb fields. In particular, the factor of $4/3$ in the electromagnetic inertial mass ($= 4U_0/3c^2$) of a spherical charge in classical electrodynamics is intimately connected with the factor of $2/3$ found both in Larmor’s formula and in the radiation-reaction formula. For long, this “mysterious” factor of $4/3$ has been considered to be undesirable and modifications in classical electromagnetic theory have been suggested to get rid of this factor (see e.g. references cited in [14, 17]). If one does adopt such a modification, then one can not see the relation between Larmor’s formula and the radiation-reaction equation. Moreover, as explicitly shown in [17], the factor of $4/3$ in the electromagnetic inertial mass has a natural explanation in the conventional electromagnetic theory when a full account is taken of all the work done by the electromagnetic forces during the process of attaining such a charge distribution.

We can generalize the formula for radiative losses, as given by the second term on the right hand side of Eq. (7), to a relativistic case, by using the condition of relativistic covariance (see e.g. ref. [2]). The excess rate of change in the electromagnetic field energy (over and above that needed for a motion of the charge with the “present” velocity) can in this way be shown to be,

$$P = -\frac{2e^2}{3c^3}\gamma^4 \left[\ddot{\mathbf{v}} \cdot \mathbf{v} + 3\gamma^2 \frac{(\dot{\mathbf{v}} \cdot \mathbf{v})^2}{c^2} \right] . \quad (10)$$

This should be contrasted with the rate implied by Liénard’s formula, where the instantaneous power radiated is supposed to be [1],

$$P = \frac{2e^2}{3c^3}\gamma^4 \left[\dot{\mathbf{v}}^2 + \gamma^2 \frac{(\dot{\mathbf{v}} \cdot \mathbf{v})^2}{c^2} \right] . \quad (11)$$

Actually the Eqs. (10) and (11) yield identical results for the instantaneous power rate in the case of a circular motion. This is to be expected since $\dot{\mathbf{v}} \cdot \mathbf{v} = \dot{\mathbf{v}}_{ret} \cdot \mathbf{v}_{ret} = 0$ in this case. Even for other periodic motions (e.g., a one-dimensional harmonic oscillation of a charge, say, within a dipole antenna) the average radiation rate over a full cycle is the same from both formulae. But for a more general case, the two formulae may not yield identical results. Specifically, in the case of a uniformly accelerated charge (i.e., for $\ddot{\mathbf{v}} + 3\gamma^2 \dot{\mathbf{v}}(\dot{\mathbf{v}} \cdot \mathbf{v})/c^2 = 0$, [5]), while Eq. (10) yields a nil rate, Eq. (11) implies a constant finite rate of radiation throughout. As is well known, the self-Coulomb field energy of a charge moving with a uniform velocity is different for different values of the velocity (see e.g., ref [17]). Now for a uniform acceleration case the sum of both terms on the right hand side of Eq. (9), as evaluated at the retarded time, represents just the rate of increase in self-Coulomb field energy of the charge as measured at the “present” time ($[\mathbf{v} \cdot \dot{\mathbf{v}} + \dot{\mathbf{v}}^2 r_0/c]_{ret} = \mathbf{v} \cdot \dot{\mathbf{v}}$, for $\dot{\mathbf{v}}_{ret} = \dot{\mathbf{v}}$). Thus the Larmor term, as calculated at the retarded time, in this case just compensates for the fact that the rate of change in the self-Coulomb field energy of a charge, as indicated by its velocity and acceleration values at the retarded time, is different from that presently required. In fact from detailed analytical calculations it indeed turns out [19, 20] that for a uniformly accelerated charge, total energy in the fields (i.e., including both the velocity and acceleration field terms from Eq. (1)) at any time t_0 is just equal to the self-energy of a charge moving uniformly with a velocity equal to the instantaneous “present” velocity of the accelerated charge (even though the detailed field configurations may differ in the two cases). There is no other excess (radiation!) energy in fields in this case. It is only in the case of a changing acceleration that the present rate of work done against self-fields (proportional to a *retarded* value of acceleration because of the time-retarded effects of the self-interaction) is different from the rate demanded by its changing self-Coulomb field energy (as calculated from the *present* value of

acceleration), this difference ultimately representing the radiated power. Since for a uniform acceleration case the retarded value of acceleration is the same as its present value, there is no radiation. It is now clear that the radiation losses for an electric charge do not merely result from its accelerated motion, but are instead a consequence of *a change in the acceleration*. Moreover it is also obvious that Larmor’s formula or its relativistic generalization, Liénard’s formula, do not unequivocally represent the radiated power losses from an accelerated charge, in the most general case.

Dirac [21] gave an alternative derivation of the radiation reaction equation that involved both retarded and advanced potentials, which is sometimes open to criticism based on the causality arguments. Moreover, from Dirac’s treatment, where the self-energy term was eliminated altogether, one can not see the relation between the radiative losses inferred from the radiation reaction equation and those implied by Larmor’s formula, without some ad hoc assumption of an acceleration dependent “bound-energy” term, as mentioned earlier. In that sense, the approach of Lorentz and Schott for calculating the radiation reaction from the self-force of a charge distribution is preferable to that of Dirac. The question of the rest-mass of an *actual* elementary charged particle lies beyond the scope of classical electrodynamics, but the divergence of the self-energy of a “point” charge (i.e., when $r_0 \rightarrow 0$) in classical electrodynamics could (at least in principle) be taken care of by invoking the presence of binding forces (needed in any case for the stability of the classical charged particle) that are equal and opposite to the electromagnetic forces of self-repulsion and assuming further that the observed rest-mass of a charged particle is a combined effect of the two. But as we have shown the concept of self-field energy can not be altogether divorced from that of radiation in a consistent classical electrodynamical approach.

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